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THE MIXED TREE DOMINATION POLYNOMIALS OF SOME DERIVED GRAPHS

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Abstract. The mixed tree domination polynomial of a connected graph G of order n is the polynomial $P(G,x) = \sum_{i=\gamma_{mt}(G)}^{2n-1} p(i)x^i$, where $p(i)$ is the number of mixed tree dominating sets of G of cardinality i and $\gamma_{mt}(G)$ is the mixed tree domination number of G. In this paper the $mtd-$ polynomial of some corona graphs of the cycle C_n and complete graph K_n are studied.

keywords:Domination, Mixed tree domination, domination polynomial, Mixed tree domination polynomial($mtd-$ polynomial)

1. Introduction

The domination polynomial of a graph is introduced by Saeid Alikhani and Yee-hock Peng in [5]. Preethi and Raji introduced the concept of mixed tree dominating set in connected graph[2,3,4]. While extending the concept of domination polynomial in view of mixed tree dominating set, we came across with many interesting relations with the coefficients of the polynomial and the graph parameters. Also, the coefficients of the polynomial of some important class of graphs have attractive patterns and the roots of the polynomial have interesting nature. This paper includes the $mtd-$ polynomial of some corona graph of C_n and K_n.

Let $G = (V, E)$ be a simple graph. For any vertex $v \in V$, the *open neighbourhood* of v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighbourhood* of v is the set $N[v] = N(v) \bigcup \{v\}$. For a set $S \subseteq V$, the *open neighbourhood* of S is $N(S) = \cup_{v\in S} N(v)$ and the *closed neighbourhood* of S is $N[S] = N(S) \cup S$. A set$S \subseteq V$is a *dominating set* of G, if $N[S] = V$, or equivalently every vertex in$V \setminus S$ is adjacent to atleast one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set with cardinality $\gamma(G)$ is called a $\gamma - set$. For the basic concepts in graph theory we refer mainly Bondy and Murthy[1] and the concepts in domination theory and mixed tree domination are followed from Preethi[3].

Saeid Alikhani and Yee-hock Peng introduced the concept of domination polynomial of a graph as the polynomial $D(G, x) = \sum_{i=\gamma(G)}^{n} d(G, i)x^i$,
where d(G,i) denotes the number of dominating sets of cardinality i . *A mixed dominating set* of G is a subset K of $V \bigcup E$ such that every element in $(V \bigcup E)\backslash K$ is either adjacent or incident to an element of K. By *the graph formed by a subset* A of $V \bigcup E$, we mean the subgraph whose edge set is $A \bigcap E$ and the vertex set consists the vertices in A together with the ends of the edges in A.

A mixed dominating set $S \subseteq V \bigcup E$ of a connected graph $G(V, E)$ is a *mixed tree dominating set*($mtd - set$) if the graph formed by S is tree.*The mixed tree domination number* $\gamma_{mt}(G)$ is the minimum cardinality of a mixed tree dominating set in G.

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Considering the polynomial idea of Alikhani et.al., we introduced the $mtd-$ polynomial[6] of a connected graph and study the information about the graph that we can obtain from the polynomial. The graphs considered here are all connected and simple of order n.

2. Mixed Tree Domination Polynomial

As the elements of a $mtd-$ set form a tree, the maximum cardinality of an $mtd-$ set is 2n-1, and the minimum cardinality is $\gamma_{mt}(G)$.

Definition 1. Let G be a connected graph of order n. The mixed tree domination polynomial of G is the polynomial $P(G,x) = \sum_{i=\gamma_{mt}(G)}^{2n-1} p(i)x^i$, where $p(i)$ is the number of mixed tree dominating sets of G of cardinality i and $\gamma_{mt}(G)$ is the mixed tree domination number of G.

If S is an $mtd-set$, as the elements of S form a tree, each vertex in S must be the end of some edge in S. So that in any $mtd-set$ S of a graph the number of edges cannot exceed $n-1$.

Example 1. Consider the path $P_3 = v_1v_2v_3$. It has only one γ_{mt} set. The mixed tree dominating sets are $\{v_2\}, \{v_1v_2, v_2v_3\}$, $\{v_1v_2, v_2\}, \{v_2, v_2v_3\}, \{v_1v_2, v_2v_3, v_1\}, \{v_1v_2, v_2v_3, v_2\}$, $\{v_1v_2, v_2v_3, v_3\}, \{v_1, v_2, v_1v_2\}$, $\{v_2, v_3, v_2v_3\}, \{v_1v_2, v_2v_3, v_1, v_2\}, \{v_1v_2, v_2v_3, v_1, v_3\}$, $\{v_1v_2, v_2v_3, v_3, v_2\}$ and $\{v_1, v_2, v_3, v_1v_2, v_2v_3\}$. So that the polynomial is

$$x + 3x^2 + 5x^3 + 3x^4 + x^5.$$

Observations[6]
The following results are immediate consequences of the definition.

Theorem 1. (1) $x^{\gamma_{mt}(G)}$ is a factor of $P(G,x)$ ie; '0' is a root of multiciplicity $\gamma_{mt}(G)$.
(2) $mtd-$ polynomial of any graph is an odd degree polynomial .
(3) The constant term is zero. The coefficient of x is 1 if and only if G is a star $K_{1,t};\ t > 1$.

Theorem 2. [6]
(1) The leading coefficient of $P(G,x)$ is $\tau(G)$ - the number of spanning trees of G.
(2) The leading coefficient is 1 ie $P(G,x)$ is monic if and only if G is tree.
(3) If G is hamiltonian then the leading coefficient is greater than or equal to n. But the converse not true.

Theorem 3. The mixed tree domination polynomial of the graph $C_n \circ K_1$ is $\sum_{k=0}^{2n}(nC_{n-1})[(nC_0)(0C_k)+(nC_1)(2C_k)+(nC_2)(4C_k)+ \ldots.+(nC_m)((2m)C_k)+\ldots..+(nC_n)((2n)C_k)]x^{2n-1+k}$.

Proof. Let $C_n = v_1v_2v_3...v_nv_1$ and let u_i, i=1,2,...,n be the vertices in $C_n \circ K_1$ adjacent to v_i. Every $mtd-$ set must contain any n-1 edges of C_n and $\gamma_{mt}(C_n \circ K_1) = 2n-1$.
Let us consider the $mtd-$ sets of cardinality 2n-1. Following are the possible sets.

- n-1 edges and n vertices: There is only one set in this category- the set consisting of all vertices and any n-1 edges of the C_n.
  $\therefore nC_{n-1}$ $mtd-$ sets in this category.
- n edges and n-1 vertices: There are $(nC_{n-1})(nC_1)$ choices for taking n edges ie, any one of the edge u_iv_i together with any n-1 edges of C_n. To dominate the remaining vertices all the remaining $v_i's$ must be included.
- In general, for an $mtd-$ set with n-1+r , $0 \le r \le n$ edges and n-r vertices: There are $(nC_{n-1})(nC_r)$ choices for edges and then the remaining vertices $v_i's$ which are not the ends of the r edges chosen must be included; so that the number of $mtd-$ sets with this specification is $(nC_{n-1})(nC_r)$.
  $\therefore$ The total number of $mtd-$ sets of cardinality 2n-1=$nC_{n-1}[1 + nC_1 + nC_2 + ... + nC_n]$

Now consider $mtd-$ sets of cardinality 2n. Following are the possible sets.

- n-1 edges and n+1 vertices not possible.
- n edges and n vertices: For n edges we have $(nC_{n-1})(nC_1)$ choices. Here n-1 vertices of C_n are compulsory. For the remaining one vertex there are $(2C_1)$ choices.
  $\therefore$ In this category there are $(nC_{n-1})(nC_1)(2C_1)$ $mtd-$sets exists.
- n+1 edges and n-1 vertices: For n+1 edges we have $(nC_{n-1})(nC_2)$ choices. In this case n-2 vertices of C_n must be included in the $mtd-$ set. For the remaining one vertex there are $4C_1$ choices.
  $\therefore$ In this category $(nC_{n-1})(nC_2)(4C_1)$ $mtd-$ sets exist.
- In general, for an $mtd-$ set with n+r edges and n-r vertices $0 \le r \le n-1$; there are $(nC_{n-1})(nC_{r+1})$ choices for edges and then the remaining vertices $v_i'S$ which are not the

ends of the r+1 edges chosen must be included; n-(r+1) vertices are compulsory. For the remaining one vertex there are $(2(r+1)C_1)$ choices.
$\therefore$ The total number of $mtd-$ sets of cardinality 2n $=(nC_{n-1})[(nC_1)(2C_1)+(nC_2)(4C_1)+ ... +(nC_n)((2n)C_1)]$

For the general case, we consider, $mtd-$ sets of cardinality 2n-1+i, $0 \leq i \leq 2n$
Case: n-1 edges and n+i vertices; here i=0 is the only possibility and nC_{n-1} such $mtd-$ sets exist.
Case: n edges and n+(i-1) vertices; i=0, i=1 and i=2 are possible.
By above case only $(nC_{n-1}(nC_1)$ choices for the n^{th} edge.
When i=0, $(nC_{n-1})(nC_1)$ such $mtd-$ sets in this category.
When i=1, $2C_1$ choices for n^{th} vertex. So in this case $(nC_{n-1}(nC_1)(2C_1)$ choices for the set.
When i=2, $(nC_{n-1})(nC_1)(2C_2)$ choices for the set.
Case: n+1 edges and (n+i-2) vertices: In this case there are $(nC_{n-1})(nC_2)$ choices for the additional two edges.
Here i can have values 0,1,2,3,4.
When i=0, only one choice for n-2 vertices; so there are $(nC_{n-1})(nC_2)$ such $mtd-$ sets.
When i=1, the additional vertex must be the end of one of the two additional edges selected, so that $4C_1$ choices for vertices, and hence $(nC_{n-1})(nC_2)(4C_1)$ choices for the set.
When i=2, the additional two vertices must be the ends of the two additional edges selected, that yeilds $4C_2$ choices for vertices, and hence $(nC_{n-1})(nC_2)(4C_2)$ $mtd-$ sets.
When i=3, there are $4C_3$ choices for vertices and so in this case $(nC_{n-1})(nC_2)(4C_3)$ $mtd-$ sets.
When i=4, there are $(nC_{n-1})(nC_2)(4C_4)$ $mtd-$ sets.
We consider the general case as follows:
(n-1)+k edges and n+i-k vertices, $0 \leq k \leq n$. Here i can have values 0,1,2,.....,2k.
When i=0 as k runs from 0 to n, we get all the $mtd-$sets of cardinality 2n-1,
which is $(nC_{n-1})[(nC_0)+(nC_1)(2C_0)+(nC_2)(4C_0)+ ...... +(nC_k)((2k)C_0)+ ......(nC_n)((2n)C_0)]$
Similarly i=1 counts the $mtd-$ sets of cardinality 2n, that is $(nC_{n-1})[(nC_1)(2C_1)+(nC_2)(4C_1)+ ...... + (nC_k)((2k)C_1)+ ......(nC_n)((2n)C_1)]$.
From the above argument we conclude that $mtd-$ sets of cardinality (2n-1)+k, $0 \leq k \leq 2n$ is given by $(nC_{n-1})[(nC_0)(0C_k)+(nC_1)(2C_k)+(nC_2)(4C_k)+ .... +(nC_m)((2m)C_k)+ ..... +(nC_n)((2n)C_k)]$.
Where $0C_0$=1 and mC_k=0, if $m < k$.

$\therefore$ $mtd-$ polynomial of $C_n \circ K_1$ is $P(C_n \circ K_1) = \sum_{k=0}^{2n}(p_{2n-1+k})x^{2n-1+k}$
Where $p_{2n-1+k}=(nC_{n-1})[(nC_0)(0C_k)+(nC_1)(2C_k)+(nC_2)(4C_k)+ .... + (nC_m)((2m)C_k)+ ..... +(nC_n)((2n)C_k)]$. □

Theorem 4. The mixed tree domination polynomial of the graph $C_n \circ K_2$ is
$\sum_{k=0}^{3n} nC_{n-1}[(2C_1)^n(2)^n(nC_k)+(2)^{n-1}(2C_1)^{n-1}(3C_2)((n+2)C_k)+(2)^{n-2}(2C_1)^{n-2}(3C_2)^2((n+4)C_k)+ ... + (2)^{n-n}(2C_1)^{n-n}(3C_2)^n((n+2n)C_k)]x^{3n-1+k}$.

Proof. Let $C_n = v_1v_2v_3...v_nv_1$ and let u_i, w_i, i = 1,2,...,n be the vertices in $P_n \circ K_2$ adjacent to v_i. Every $mtd-$ set must contain any n-1 edges of C_n and one edge from each pair $\{v_iu_i, v_iw_i\}$ and $\gamma_{mt}(C_n \circ K_2) = 3n-1$.
For convenience the blocks $<\{v_i, u_i, w_i\}>$, i=1,2,...,n are called the 3-blocks.
Let us consider the $mtd-$ sets of cardinality **3n+i-1**, $0 \leq i \leq 3n$
Case: i=0; we consider the $mtd-$ sets of cardinality 3n-1. Following are the possible sets.

(1) 2n-1 edges and n vertices:
For 2n-1 edges we have $(nC_{n-1})(2C_1)^n$ choices, by the above arguments. Let the 2n-1 edges be any n-1 edges of C_n and $v_1x_1, v_2x_2, ..., v_nx_n$, where $x_i \in \{u_i, w_i\}$. Then for the n vertices, either v_i or x_i can be chosen from each block; so that there are $(nC_{n-1})(2C_1)^n2^n$ $mtd-$ sets in this category.

(2) 2n edges and n-1 vertices:
For 2n edges we have $(nC_{n-1})(2C_1)^{n-1}(3C_2)$ choices; because, as we cannot include all the three edges of a block, we must choose the 2n-1 edges as above and the additional edge can be any one of v_iy_i or u_iw_i where $y_i \in \{u_i, w_i\}$. That is 2 edges from one of the block $<\{v_i, u_i, w_i\}>$ and one edge from each pair $\{v_ju_j, v_jw_j\}$, $j \neq i$; that yeilds 2^{n-1} choices for the n-1 vertex.
$\therefore$ In this case $(nC_{n-1})(2C_1)^{n-1}(3C_2)(2^{n-1})((n+2)C_0)$ $mtd-$ sets exist.

(3) In general, for an $mtd-$ set with 2n-1+r, $0 \leq r \leq n$ edges and n-r vertices:
There are $(nC_{n-1})(2)^{n-r}(2C_1)^{n-r}(3C_2)^r$

$((n+2r)C_0)$ $mtd-$ sets.

$\therefore$ In this case $(nC_{n-1})[(2)^n(2C_1)^n(nC_0)+(2)^{n-1}(2C_1)^{n-1}(3C_2)((n+2)C_0)+(2)^{n-2}(2C_1)^{n-2}(3C_2)^2((n+4)C_0)+...+(2)^{n-n}(2C_1)^{n-n}(3C_2)^n((n+2n)C_0)]$ $mtd-$ sets exist.

Case: i=1; we consider the $mtd-$ sets of cardinality 3n. Following are the possible sets.

(1) 2n-1 edges and n+1 vertices:
As in the above case, there are $(nC_{n-1})(2C_1)^n$ choices for 2n-1 edges, 2^n choices for n vertices, which are compulsory and nC_1 choices for the additional vertex. So that $(nC_{n-1})(2C_1)^n(2^n)(nC_1)$ $mtd-$sets in this category.

(2) 2n edges and n vertices:
For 2n edges we have, as above $(nC_{n-1})(2C_1)^{n-1}(3C_2)$ choices. For n vertices we have $(2C_1)^{n-1})$ $(n+2)C_1$ choices because one vertex must be taken from each of the n-1 blocks from which only one edge say v_ix_i is taken; one end of the edge v_ix_i must be included. So 2^{n-1} choices for n-1 vertices and the remaining vertex has $(n+2)C_1$ choices.
$\therefore$ In this category we have $(nC_{n-1})(2C_1)^{n-1}(3C_2)2^{n-1}$ $(n+2)C_1$ $mtd-$ sets.

(3) In general, for an $mtd-$ set with 2n-1+r, $0 \leq r \leq n$ edges and n-r+1 vertices: For 2n-1+r edges any n-1 edges of C_n and $v_1x_1,...,v_nx_n$ are compulsory; (as above); where $x_i \in \{u_i, w_i\}$. And the remaining r edges must be taken one from each r blocks. So that two edges are selected from each of the r 3-blocks; and one from each of the remaining 3-blocks; so that $(nC_{n-1})(2C_1)^{n-r}(3C_2)^r$ choices for 2n-1+r edges.
Now, for n-r+1 vertices, we must choose one end of the edge v_ix_i from the n-r 3-blocks from which only one edge is included; that gives 2^{n-r} choices. The remaining vertex has 3r+n-r=n+2r choices; that is $(n+2r)C_1$(See the graph given below). So that, the number of $mtd-$ sets in this category is $(nC_{n-1})(2)^{n-r}(2C_1)^{n-r}$ $(3C_2)^r((n+2r)C_1)$.

$\therefore$ the number of $mtd-$ sets of cardinality 3n is $(nC_{n-1})[(2C_1)^n(2)^n(nC_1)+(2)^{n-1}(2C_1)^{n-1}(3C_2)((n+2)C_1)+(2)^{n-2}(2C_1)^{n-2}(3C_2)^2((n+4)C_1)+...+(2)^{n-n}(2C_1)^{n-n}(3C_2)^n((n+2n)C_1)]$.

Case: i=k, $0 \leq k \leq 3n$

(1) 2n-1 edges and n+k vertices: In this case $k \leq n$.
2n-1 edges has $(nC_{n-1})(2C_1)^n$ choices, n vertices has 2^n choices and nC_k choices for the remaining k vertices; gives $(nC_{n-1})(2C_1)^n(2^n)(nC_k)$ choices.

(2) 2n edges and n+k-1 vertices; $k \leq n+2$.
As above 2n edges has $(nC_{n-1})(2C_1)^{n-1}(3C_2)$ choices; 2^{n-1} choices for n-1 vertices from the n-1 blocks from which only one edge is taken, and $(n+2)C_k$ (n+2=n-1+3) choices for k vertices so that $(nC_{n-1})(2C_1)^{n-1}(3C_2)(2^{n-1})$ $(n+2)C_k$
.
.
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(3) 3n-1 edges, k vertices $k \leq 3n = n+2n$.
There are $(3C_2)^n$ choices for 3n-1 edges and $(n+2n)C_k$ =$3nC_k$ choices for k vertices.

$\therefore$ the number of $mtd-$ sets of cardinality 3n+k-1 is $(nC_{n-1})[(2C_1)^n(2)^n(nC_k)+(2)^{n-1}(2C_1)^{n-1}(3C_2)((n+2)C_k)+(2)^{n-2}(2C_1)^{n-2}(3C_2)^2((n+4)C_k)+...+(2)^{n-n}(2C_1)^{n-n}(3C_2)^n((n+2n)C_k)]$.

$\therefore$ $mtd-$ polynomial of $C_n \circ K_2$ is $p(C_n \circ K_2) = \sum_{k=0}^{3n}(p_{3n-1+k})x^{3n-1+k}$

Where $p_{3n-1+k} = (nC_{n-1})[(2C_1)^n(2)^n(nC_k)+(2)^{n-1}(2C_1)^{n-1}(3C_2)((n+2)C_k)+(2)^{n-2}(2C_1)^{n-2}(3C_2)^2((n+4)C_k)+...+(2)^{n-n}(2C_1)^{n-n}(3C_2)^n((n+2n)C_k)]$.

□

Theorem 5. The mixed tree domination polynomial of the graph $K_n \circ K_1$ is $\sum_{k=0}^{2n}\tau(K_n)[(nC_0)(0C_k)+(nC_1)(2C_k)+(nC_2)(4C_k)+....+(nC_m)((2m)C_k)+.....+(nC_n)((2n)C_k)]x^{2n-1+k}$, where $\tau(K_n) = n^{n-2}$.

Proof. Let $K_n = v_1v_2v_3...v_n$ and let u_i, i=1,2,...,n be the vertices in $K_n \circ K_1$ adjacent to v_i. Every $mtd-$ set must contain any n-1 edges of K_n and $\gamma_{mt}(K_n \circ K_1) = 2n-1$.
Let us consider the $mtd-$ sets of cardinality 2n-1. Following are the possible sets.

- n-1 edges and n vertices: Since the n-1 edges of K_n form a spanning tree, the number of ways n-1 edges can be choosen is the number of spanning trees of K_n that is $\tau(K_n) = n^{n-2}$. $\therefore n^{n-2}$ mtd-sets exists in this category.

- n edges and n-1 vertices: There are $n^{n-2}(nC_1)$ choices for taking n edges ie, any one of the edge u_iv_i together with any n-1 edges of K_n. To dominate the remaining vertices all the remaining $v_i's$ must be included.
- In general, for an $mtd-$ set with n-1+r ,$0 \leq r \leq n$ edges and n-r vertices: There are $n^{n-2}(nC_r)$ choices for edges and then the remaining vertices $v_i's$ which are not the ends of the r edges chosen must be included; so that the number of $mtd-$ sets with this specification is $n^{n-2}(nC_r)$.
$\therefore$ The total number of $mtd-$ sets of cardinality 2n-1=$n^{n-2}[1 + nC_1 + nC_2 + ... + nC_n]$

Now consider $mtd-$ sets of cardinality 2n. Following are the possible sets.

- n-1 edges and n+1 vertices not possible.
- n edges and n vertices: For n edges we have $n^{n-2}(nC_1)$choices. Here n-1 vertices of K_n are compulsory. For the remaining one vertex there are $(2C_1)$ choices.
$\therefore$ In this category there are $n^{n-2}(nC_1)(2C_1)$ $mtd-$sets exists.
- n+1 edges and n-1 vertices: For n+1 edges we have $n^{n-2}(nC_2)$ choices. In this case n-2 vertices of K_n must be included in the $mtd-$ set. For the remaining one vertex there are $4C_1$ choices.
$\therefore$ In this category $n^{n-2}(nC_2)(4C_1)$ $mtd-$ sets exist.
- In general, for an $mtd-$ set with n+r edges and n-r vertices $0 \leq r \leq n-1$;
there are $n^{n-2}(nC_{r+1})$ choices for edges and then the remaining vertices $v_i'S$ which are not the ends of the r+1 edges chosen must be included; n-(r+1) vertices are compulsory. For the remaining one vertex there are $(2(r+1)C_1)$ choices.
$\therefore$ The total number of $mtd-$ sets of cardinality 2n $=n^{n-2}[(nC_1)(2C_1) + (nC_2)(4C_1) + ... + (nC_n)((2n)C_1)]$

For the general case, we consider, $mtd-$ sets of cardinality 2n-1+i, $0 \leq i \leq 2n$
Case: n-1 edges and n+i vertices; here i=0 is the only possibility and n^{n-2} such $mtd-$ sets exist.
Case: n edges and n+(i-1) vertices; i=0, i=1 and i=2 are possible.
By above case only $n^{n-2}(nC_1)$ choices for the n^{th} edge.
When i=0, $n^{n-2}(nC_{n-1}$ such $mtd-$ sets in this category.
When i=1, $2C_1$ choices for n^{th} vertex. So in this case $n^{n-2}(nC_1)(2C_1)$ choices for the set.
When i=2, $n^{n-2}(nC_1)(2C_2)$ choices for the set.
Case: n+1 edges and (n+i-2) vertices: In this case there are $n^{n-2}(nC_2)$ choices for the additional two edges.

Here i can have values 0,1,2,3,4.
When i=0, only one choice for n-2 vertices; so there are $n^{n-2}(nC_2)$ such $mtd-$ sets.
When i=1, the additional vertex must be the end of one of the two additional edges selected, so that $4C_1$ choices for vertices, and hence $n^{n-2}(nC_2)(4C_1)$ choices for the set.
When i=2, the additional two vertices must be the ends of the two additional edges selected, that yeilds $4C_2$ choices for vertices, and hence $n^{n-2}(nC_2)(4C_2)$ $mtd-$ sets.
When i=3, there are $4C_3$ choices for vertices and so in this case $n^{n-2}(nC_2)(4C_3)$ $mtd-$ sets.
When i=4, there are $n^{n-2}(nC_2)(4C_4)$ $mtd-$ sets.
We consider the general case as follows:
(n-1)+k edges and n+i-k vertices, $0 \leq k \leq n$. Here i can have values 0,1,2,.....,2k.
When i=0 as k runs from 0 to n, we get all the $mtd-$sets of cardinality 2n-1,
which is $n^{n-2}[(nC_0) + (nC_1)(2C_0) + (nC_2)(4C_0) + ... + (nC_k)((2k)C_0) + ...(nC_n)((2n)C_0)]$
Similarly i=1 counts the $mtd-$ sets of cardinality 2n, that is $n^{n-2}[(nC_1)(2C_1) + (nC_2)(4C_1) + ... + (nC_k)((2k)C_1) + ...(nC_n)((2n)C_1)]$.
From the above argument we conclude that $mtd-$ sets of cardinality (2n-1)+k, $0 \leq k \leq 2n$ is given by $n^{n-2}[(nC_0)(0C_k) + (nC_1)(2C_k) + (nC_2)(4C_k) + ... + (nC_m)((2m)C_k) + ... + (nC_n)((2n)C_k)]$.
Where $0C_0$=1 and mC_k=0, if $m < k$.
$\therefore$ $mtd-$ polynomial of $K_n \circ K_1$ is $P(K_n \circ K_1) = \sum_{k=0}^{2n} (p_{2n-1+k})x^{2n-1+k}$
Where $p_{2n-1+k} = n^{n-2}[(nC_0)(0C_k) + (nC_1)(2C_k) + (nC_2)(4C_k) + ... + (nC_m)((2m)C_k) + ... + (nC_n)((2n)C_k)]$. □

Theorem 6. The mixed tree domination polynomial of the graph $K_n \circ K_2$ is

$$\sum_{k=0}^{3n} n^{n-2}[(2C_1)^n(2)^n(nC_k) + (2)^{n-1}(2C_1)^{n-1}(3C_2)((n+2)C_k) + (2)^{n-2}(2C_1)^{n-2}(3C_2)^2((n+4)C_k) + ... + (2)^{n-n}(2C_1)^{n-n}(3C_2)^n((n+2n)C_k)]x^{3n-1+k}$$

Proof. Let $K_n = v_1v_2v_3...v_n$ and let u_i, w_i, i = 1,2,...,n be the vertices in $K_n \circ K_2$ adjacent to v_i. Every $mtd-$ set must contain any n-1 edges of K_n and one edge from each pair $\{v_iu_i, v_iw_i\}$ and $\gamma_{mt}(K_n \circ K_2) = 3n-1$. For convenience the blocks $< \{v_i, u_i, w_i\} >$, i=1,2,...,n are called the 3-blocks.
Let us consider the $mtd-$ sets of cardinality **3n+i-1**, $0 \leq i \leq 3n$
Case: i=0; we consider the $mtd-$ sets of cardinality 3n-1. Following are the possible sets.

(1) 2n-1 edges and n vertices:
For 2n-1 edges we have $n^{n-2}(2C_1)^n$ choices, by the above arguments. Let the 2n-1 edges be any n-1 edges of K_n, it has $\tau(K_n) = n^{n-2}$ choices and $v_1x_1, v_2x_2, ..., v_nx_n$, where $x_i \in \{u_i, w_i\}$. Then for the n vertices, either v_i or x_i can be chosen from each block; so that there are $n^{n-2}(2C_1)^n2^n$ $mtd-$ sets in this category.

(2) 2n edges and n-1 vertices:
For 2n edges we have $n^{n-2}(2C_1)^{n-1}(3C_2)$ choices; because, as we cannot include all the three edges of a block, we must choose the 2n-1 edges as above and the additional edge can be any one of v_iy_i or u_iw_i where $y_i \in \{u_i, w_i\}$. That is 2 edges from one of the block $< \{v_i, u_i, w_i\} >$ and one edge from each pair $\{v_ju_j, v_jw_j\}$, $j \neq i$; that yeilds 2^{n-1} choices for the n-1 vertex.
$\therefore$ In this case $n^{n-2}(2C_1)^{n-1}(3C_2)(2^{n-1})$ $((n+2)C_0)$ $mtd-$ sets exist.

(3) In general, for an $mtd-$ set with 2n-1+r, $0 \leq r \leq n$ edges and n-r vertices:
There are $n^{n-2}(2)^{n-r}(2C_1)^{n-r}(3C_2)^r((n+2r)C_0)$ $mtd-$ sets.

$\therefore$ In this case $n^{n-2}[(2)^n(2C_1)^n(nC_0)+(2)^{n-1}(2C_1)^{n-1}(3C_2)$ $((n+2)C_0)+(2)^{n-2}(2C_1)^{n-2}(3C_2)^2((n+4)C_0)+...+$ $(2)^{n-n}(2C_1)^{n-n}(3C_2)^n((n+2n)C_0)]$ $mtd-$ sets exist.
Case: i=1; we consider the $mtd-$ sets of cardinality 3n. Following are the possible sets.

(1) 2n-1 edges and n+1 vertices:
As in the above case, there are $n^{n-2}(2C_1)^n$ choices for 2n-1 edges, 2^n choices for n vertices, which are compulsory and nC_1 choices for the additional vertex. So that $n^{n-2}(2C_1)^n(2^n)(nC_1)$ $mtd-$sets in this category.

(2) 2n edges and n vertices:
For 2n edges we have, as above $n^{n-2}(2C_1)^{n-1}$ $(3C_2)$ choices. For n vertices we have $(2C_1)^{n-1}$ $(n+2)C_1$ choices because one vertex must be taken from each of the n-1 blocks from which only one edge say v_ix_i is taken; one end of the edge v_ix_i must be included. So 2^{n-1} choices for n-1 vertices and the remaining vertex has $(n+2)C_1$ choices. $\therefore$ In this category we have $n^{n-2}(2C_1)^{n-1}(3C_2)2^{n-1}(n+2)C_1$ $mtd-$ sets.

(3) In general, for an $mtd-$ set with 2n-1+r, $0 \leq r \leq n$ edges and n-r+1 vertices: For 2n-1+r edges any n-1 edges of K_n and $v_1x_1, ..., v_nx_n$ are compulsory; (as above); where $x_i \in \{u_i, w_i\}$. And the remaining r edges must be taken one from each r blocks. So that two edges are selected from each of the r 3-blocks; and one from each of the remaining 3-blocks; so that $n^{n-2}(2C_1)^{n-r}(3C_2)^r$ choices for 2n-1+r edges. Now, for n-r+1 vertices, we must choose one end of the edge v_ix_i from the n-r 3-blocks from which only one edge is included; that gives 2^{n-r} choices. The remaining vertex has 3r+n-r=n+2r choices; that is $(n+2r)C_1$(See the graph given below). So that, the number of $mtd-$ sets in this category is $n^{n-2}(2)^{n-r}$ $(2C_1)^{n-r}(3C_2)^r((n+2r)C_1)$.

$\therefore$ the number of $mtd-$ sets of cardinality 3n is $n^{n-2}[(2C_1)^n(2)^n(nC_1) + (2)^{n-1}(2C_1)^{n-1}(3C_2)$ $((n+2)C_1)+(2)^{n-2}(2C_1)^{n-2}(3C_2)^2((n+4)C_1)+...+$ $(2)^{n-n}(2C_1)^{n-n}(3C_2)^n((n+2n)C_1)]$.
Case: i=k, $0 \leq k \leq 3n$

(1) 2n-1 edges and n+k vertices: In this case $k \leq n$.
2n-1 edges has $n^{n-2}(2C_1)^n$ choices, n vertices has 2^n choices and nC_k choices for the remaining k vertices; gives $n^{n-2}(2C_1)^n(2^n)(nC_k)$ choices.

(2) 2n edges and n+k-1 vertices; $k \leq n+2$.
As above 2n edges has $n^{n-2}(2C_1)^{n-1}(3C_2)$ choices; 2^{n-1} choices for n-1 vertices from the n-1 blocks from which only one edge is taken, and $(n+2)C_k$ (n+2=n-1+3) choices for k vertices so that $n^{n-2}(2C_1)^{n-1}(3C_2)(2^{n-1})$ $(n+2)C_k$
.
.
.

(3) 3n-1 edges, k vertices $k \leq 3n = n + 2n$.
There are $(3C_2)^n$ choices for 3n-1 edges and $(n+2n)C_k$ $=3nC_k$ choices for k vertices.

$\therefore$ the number of $mtd-$ sets of cardinality 3n+k-1 is
$n^{n-2}[(2C_1)^n(2)^n(nC_k) + (2)^{n-1}(2C_1)^{n-1}(3C_2)((n+2)C_k)+(2)^{n-2}(2C_1)^{n-2}(3C_2)^2((n+4)C_k)+...+(2)^{n-n}(2C_1)^{n-n}(3C_2)^n((n+2n)C_k)]$.

$\therefore$ $mtd-$ polynomial of $K_n \circ K_2$ is $p(K_n \circ K_2) = \sum_{k=0}^{3n}(p_{3n-1+k})x^{3n-1+k}$

Where $p_{3n-1+k} = n^{n-2}[(2C_1)^n(2)^n(nC_k)+(2)^{n-1}(2C_1)^{n-1}(3C_2)((n+2)C_k)+(2)^{n-2}(2C_1)^{n-2}(3C_2)^2((n+4)C_k)+...+(2)^{n-n}(2C_1)^{n-n}(3C_2)^n((n+2n)C_k)]$.

□

3. Conclusion

Our studies on $mtd-$ polynomial of corona of a graph with another graphs, is not easy to handle for general graphs. But for certain class of graphs it seems interesting. We are trying to relate the $mtd-$ polynomial of $G \circ H$ with that of G and H, where the first can be obtained. We are also trying to get an algorithm for finding the $mtd-$ polynomial of a graph G using graphs constructed from G as in the case of chromatic polynomial where $\chi(G, x) = \chi(G \setminus e, x) - \chi(G/e, x)$. We have also observed some interesting nature of the roots of the $mtd-$ polynomials of some graphs.

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